# On the use of ‘reaction-diffusion’ model in the business cycle analysis

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## Introduction

The goal of this work is to find a practical method of market condition analysis. Numerous researches are subjugated to this topical subject[1,2,3], although it would be premature to speak about their classical condition in this course.

A methodological problem faced by the market analysis is well-known and comes to the global equilibrium problem. The market is a peculiar mathematical modeling object in terms of the fact that its equilibrium conditions can be multiple. For example, a number of equilibrium conditions for a standard Arrow—Debreu model[4] comes to be infinite, and not only is it infinite but also forming a continual set[5]. Practically, it means that a condition of price development process about a local equilibrium poorly depends on the prehistory of such a process. Therefore, an import of the forecasting methodology from physical system analysis, where dependence of the current condition of the process on its prehistory is significant, to market condition analysis, occurs to be useless. Particularly, considering a character of differences between the business cycle and the cycle of lunisolar eclipses, the classical methods, excellent for analyzing the celestial mechanics cycles, do not allow us to achieve even imperfect results in the market analysis.

This research leans upon the only assumption that a mechanism to be designated as a *strongly dissipative system* produces the market condition data.

Finally, we will get all results of our research in the form of a regression on the assumption, and all our solutions will result from the analysis of the market condition data. We will use Dow-Jones Industrial Average (DJIA) time series, which are accessible data of the market condition and can be easily found on the Internet.

## Business Cycle Model

Let’s consider square self-mapping *f* of an interval $x \in [0,1]$

(1) $$f(x) = ax - bx^2 = \mu x(1-x),$$

presented on the following graph (Fig. 1) for $\mu = 0{,}8$.

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[1] Candelon Bertrand, Hecq Alain (2002) *Short-run attractor regimes and the cyclical behavior of output and prices* // Proceedings of the International Conference on Policy Modeling Université Libre de Bruxelles, July 4–6, 2002.

[2] Bayar Ali, Candelon Bertrand (2002). *Entry and exit dynamics in business cycles* // ibid.

[3] Guerrero Guillaume (2002) *Markov-switching models of business cycle: can the econometric model detect the growth regime?* // ibid.

[4] Arrow K.J., Debreu G. (1954) *Existence of equilibrium for a competitive economy* // Econometrica, **25**, 265–290.

[5] Balasko Yves, Cass D. (1989) *The structure of financial equilibrium with exogenous yields: The case of incomplete markets* // Econometrica, **57**, 135–163.

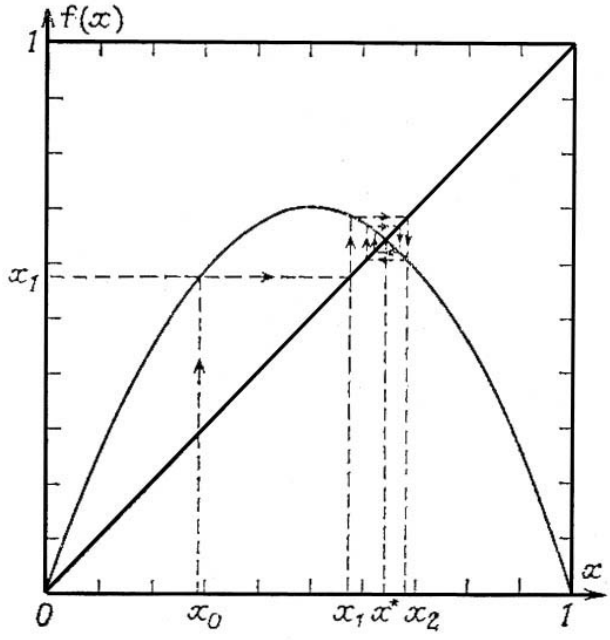


Fig. 1. Graph of *f(x)* mapping for $x \in$ *[0, 1]* interval, where $\mu$*=0,8*.

It is easy to see on Fig. 1 that the points with abscissas $x_1^*$ and $x_2^*$ (the fixed points of *f* mapping), where μ is a set point, are unstable and that a pair of values $x_1^*$ and $x_2^*$, with any entry condition $x_0$ from an open interval [0, 1], serves as an asymptotic limit of iterations, with which the images of the initial point will alternately coincide. The two indicated points form an attractor with a period of 2, which is also named a *2-cycle*. The *f(x)* maps the $x_1^*$ point into the $x_2^*$ point, and vice versa.

Since

(2) $$x_2^* = f(x_1^*) = f(f(x_2^*)) \text{ and}$$

(2′) $$x_1^* = f(x_2^*) = f(f(x_1^*)),$$

then these two points, which are not the fixed points of the *f* mapping (as it was stressed before), are the fixed points of the mapping

(3) $$g(x) = f(f(x)) = f^2(x),$$

presented on the graph for the same value of μ (Fig. 2).

A transition from the situation when an attractor with a period of 2 appears instead of that with a period of 1 (i. e. when a period doubles) takes place when the value of μ increases and the only value of $\mu_1 = 0$. At that moment, the fixed stable point of *f* mapping becomes unstable and the two fixed points of $f^2$ mapping appear correspondingly. This mapping has four fixed points, two of which ($x_1^*$ and $x_2^*$) are stable. The two squares outlining the fixed points (Fig. 2) are to stress the presence of the respective cycles around the both.

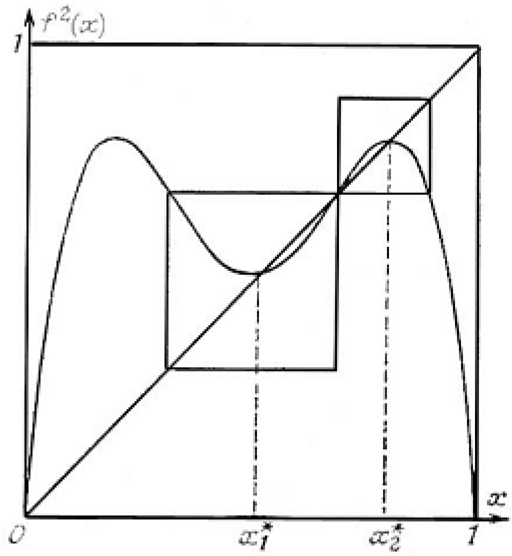


Fig. 2. Graph of *g(x)=f²(x)* mapping for $x \in [0, 1]$ interval, where $\mu=0,8$.

It is well known from the chaos theory[6] what happens when we further increase the μ value. The *f* и $f^2$ curves are being gradually deformed the way that, as a result of it, the fixed points of the $f^2$ mapping also become unstable. We can locally see a parabolic curve with a fixed stable point inside each square on Fig. 2, i. e. the cycle of speculations repeats the same logic: the fixed point of *g* mapping is substituted with the two fixed points of the $h(x) = g(g(x)) = f^4(x)$ function. The same logic can also be applied to the two fixed points of *g* mapping with $x_1^*$ and $x_2^*$ abscissas, which simultaneously become unstable when

$$\mu_2 = \frac{1+\sqrt{6}}{4} = 0{,}86237...$$

And so forth: we receive an endless cascade of bifurcations, and each one of them is accompanied by the period doubling related to the sub-harmonic instability. However, according to the Ruelle—Takens—Newhouse scenario[7], quite a few number of bifurcations could be enough for an emergence of a chaotic behavior of the system. According to this theory, one could expect that the power spectrum of a dynamic system described by the mapping of this kind would evolve into μ functions as follows: first the system power spectrum will contain one frequency ($\omega_1$), then two ones ($\omega_1$ and $\omega_2$), and sometimes three ones ($\omega_1$, $\omega_2$, $\omega_3$). As soon as the third frequency emerges in the spectrum, the noise component appears, which is characteristic for the chaos. In a practical sense, considering a system working ‘at chaos edge,’ like the market, let’s assume, for our purposes, that only two frequencies will be enough in the power spectrum for the process model.

[6] Berge P., Pomeau Y., Vidal C. *L’orde dans le chaos*, Paris: Hermann, 1988.
[7] Ibid.

# Selection of time series modes with the use of a ‘reaction-diffusion’ system

A structure of recognizing systems based on the use of dissipative dynamics is known[8]. Their essence is to relax to a condition described by a specific function. This function, being essentially analogous to power, makes a ‘potential landscape’ in the system state space. In a majority of practical cases, that ‘landscape’ has a complex form with numerous minima divided by potential barriers. Let’s consider that our recognizing system is organized the way that its different stable conditions, corresponding to the patterns, are the minima of the power function, provided that the mode, being a certain dynamic variable, corresponds to each pattern the system is to recognize. The process of recognition will imply that the system reaches the minimum corresponding to the nearest pattern to the one being analyzed.

The analogue recognition of the patterns is nowise the only method of this sort. The system dynamics may have an evolving character as well, i. e. it may base on the mode competition. It is only the mode corresponding to the nearest pattern that can ‘survive’ in the course of the system evolvement. This is an analogue to Darwin selection. Usually patterns are entered into the system parameters. The pattern being analyzed can be presented either through forming a corresponding entry condition or through modifying the system parameters. In the first case, the analogue solving of a recognition problem is based on the system dynamic regulation under its entry conditions: the depicting point occurs to be in some attractor’s basin within the state space.

Both ways of the system regulation are possible in evolving models: we will survey the parametric way of such regulation below. Let’s take some universal ‘reaction-diffusion’ model as a base, which model can be reduced to a system of Lotka—Volterra differential equations within the limit of complete interfusion and can create a series of comforts. The evolving model analyzed has a universal character in principle. However, to be evident, let’s discuss a classical form of the ‘reaction-diffusion’ model in its biochemical interpretation.

Half a century ago Alan Turing proposed the now famous reaction-diffusion system involving two chemicals[9] to model biological pattern formation, and ‘morphogenesis’. Since then, it has been extensively used in studying various speciec problems in mathematical biology[10]. In its general form the Turing system for modelling the evolution of the concentrations of two chemicals is given as

$$\begin{cases} \dfrac{\partial U}{\partial t} = D_u \nabla^2 U + f(U,V) \\ \dfrac{\partial V}{\partial t} = D_v \nabla^2 V + g(U,V) \end{cases} \quad (4)$$

[8] Bongard J.C. and Pfeifer R. (2001) Repeated Structure and Dissociation of Genotypic and PhenotypicComplexity in Artificial Ontogeny, in L. Spector et al (eds.), Proceedings of The Genetic and Evolutionary Computation Conference, GECCO-2001, San Francisco, CA, pp.829-836.

[9] Turing A.M. (1952) *The chemical basis of morphogenesis //* Phil. Trans. Roy. Soc. Lond., **B237**, 37-72.

[10] Murray J.D. Mathematical Biology, 2nd. ed. – Berlin: Springer Verlag, 1993.

where $U \equiv U(\vec{x},t)$ and $V \equiv V(\vec{x},t)$ are the unknown concentrations, and $D_u$ and $D_v$ the respective diffusion constants. The reactions are modeled by the functions f and g which are typically non-linear. Turing formalised this idea to abstract chemicals in an environment. To model this it is necessary to describe the concentrations of these chemicals over space and time. The interactions between the chemicals can then be seen as functions applied to the current concentrations. Using such a system it is possible by means of diffusion and reaction to generate very diverse pattern formation systems. The name of this type of pattern formation system is reaction-diffusion system, as the two key elements in the pattern formation are the long-range effect of diffusion and purely local reaction interactions. Diffusion is the relatively slow mechanism employed by nature to equalise the concentrations of a chemical over space. Reaction is a much faster, and specifically local effect between two chemicals. When we combine these simple mechanisms and choose the parameters of the reaction and diffusion right, it is possible to simulate and describe many (dynamic or static) pattern formation mechanisms.

With a reaction-diffusion system, we consider the dispersing two-species Lotka—Volterra model with temporally periodic intermittence of interspecific competitive relationships. We assume that the competition coefficient becomes a given positive constant and zero by turns periodically in time.

(5)
$$\begin{cases} \dfrac{\partial U}{\partial t} = D_u \dfrac{\partial^2 U}{\partial x^2} + (r_u - \alpha_u U - \beta_{uv} V) \cdot U \\ \dfrac{\partial V}{\partial t} = D_v \dfrac{\partial^2 V}{\partial x^2} + (r_v - \alpha_v V - \beta_{vu} U) \cdot V \end{cases}$$

where $U \equiv U(x,t)$ and $V \equiv V(x,t)$ is the population density of both species at position $x$ and time $t$. Parameters $D_u$ and $D_v$, $r_u$ and $r_v$, $\alpha_u$ and $\alpha_v$ are all positive, which mean respectively the diffusion coefficient to indicate the mobility, the intrinsic growth rate to give the maximal reproductive capacity, and the intra-specific competition coefficient to indicate the strength of density effect from the other individuals of same species. Functions of time $\beta_{uv} \equiv \beta_{uv}(t)$ and $\beta_{vu} \equiv \beta_{vu}(t)$ introduce the inter-specific competitive interaction between populations of species 1 and 2. These functions is now de.ned as regorously periodic in time: $\beta_{uv}(t+T) = \beta_{uv}(t)$ for any $t \geq 0$ with a given positive constant $T$.

Then, as one can show, in the presence of the 'best' several modes, the formed system is to select accidentally one of the patterns corresponding to those modes (depending on the fluctuations of all the modes' initial concentrations).

The classical scalar form of Lotka—Volterra equation looks as follows[11]:

(6) $$x(t+1) = (1 + birth - death) * x(t) - competition * x^2(t).$$

Switching to such designations as $a = 1 + birth - death, b = competition$, implying that $x$ depends on $t$, and admitting that $x \in [0,1]$, we receive a formula analogous to the self-mapping of a square interval, see (1).

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[11] Sorin S. (1997) *Generalized Lotka—Volterra (GLV) models and generic emergence of scaling laws in stock markets*. – Budapest: Econophysics (Kluver Academic Press); eds. Imre Kondor and Janos Kertes.

A minimum deviation of the approximating model from a preset DJIA time series (in terms of root-mean-square proximity) is implied here as the best approximation. Using the terms of the described recognition principle, the received approximating model is the mode having a maximum selective value.

Naturally, it does not guarantee the model from its inadequacy to the real DJIA process caused by the market, at least on long-term intervals and with equal exactness. Generally speaking, before we start discussing the received real data, one should reconsider the very assumptions, since they were taken in the interests of the model rather than of interpreting the real data, which are, in fact, strongly different from the model.

## Discussion of the data

As an example of a DJIA time series, let's consider the results of applying the described mode selection principle with the use of 'reaction-diffusion' systems. In accordance with both the model and the algorithm described above, a mode depicted on Fig. 3 was selected from the indicated time series.

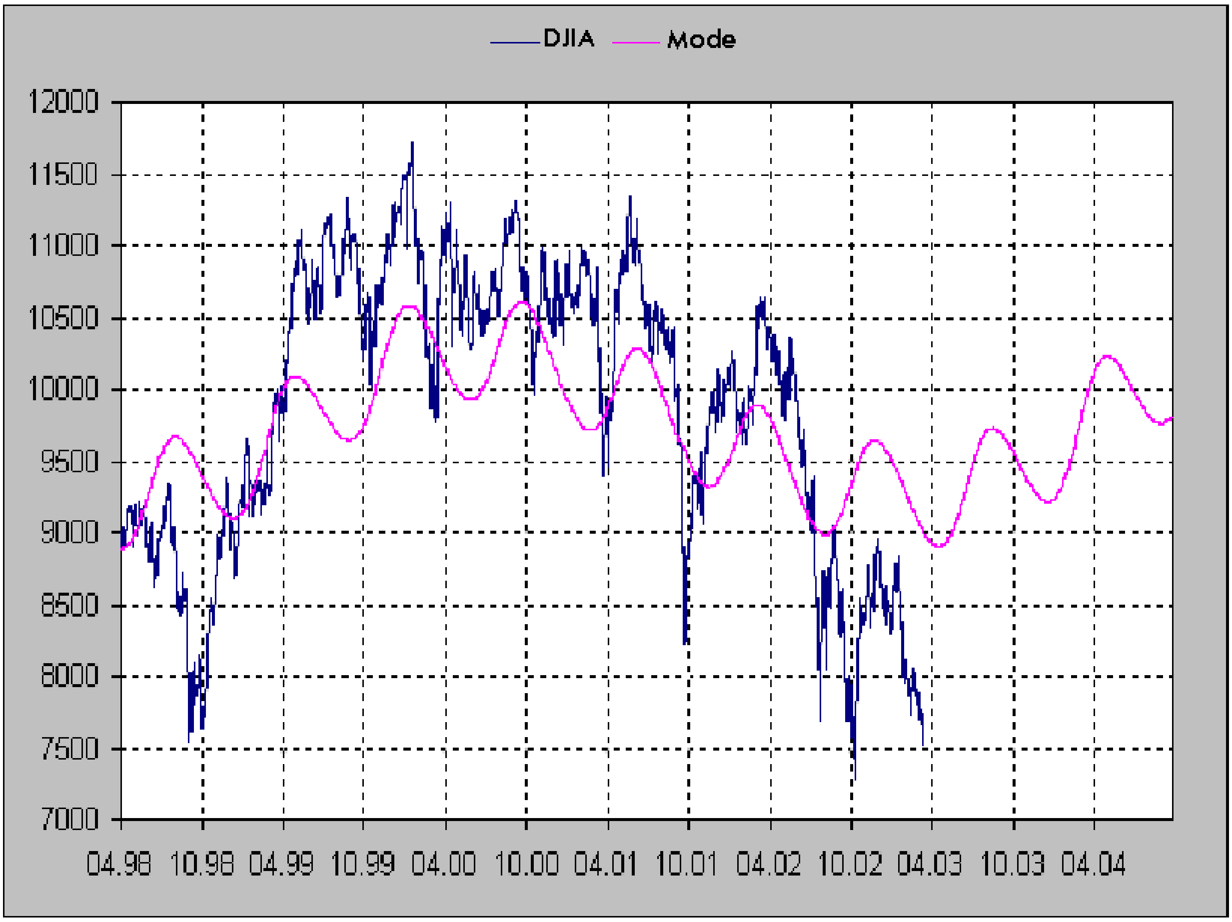


Fig. 3. Result of the mode selection from DJIA time series.

The main advantage of the method described above is its robustness. The latter reveals itself in the fact that no additions of new data to the DJIA time series, both from the 'right' and from the 'left', change the mode. For example, changes of the mode provided by the DJIA data update for the last three months are virtually within the thickness of the line on the graph.

Another advantage of the selected mode is that it is quite a successful predictive model. For example, local extrema of this curve, which could be observed in reality during several months, meet the changes in the market sentiment quite exactly. Following the graph on Fig. 4, a moderately increasing tendency, which received its essential development recently but had been forecasted on the data that were known in March, is particularly obvious. However, in relation to this solution, let's note the business cycle estimation robustness.

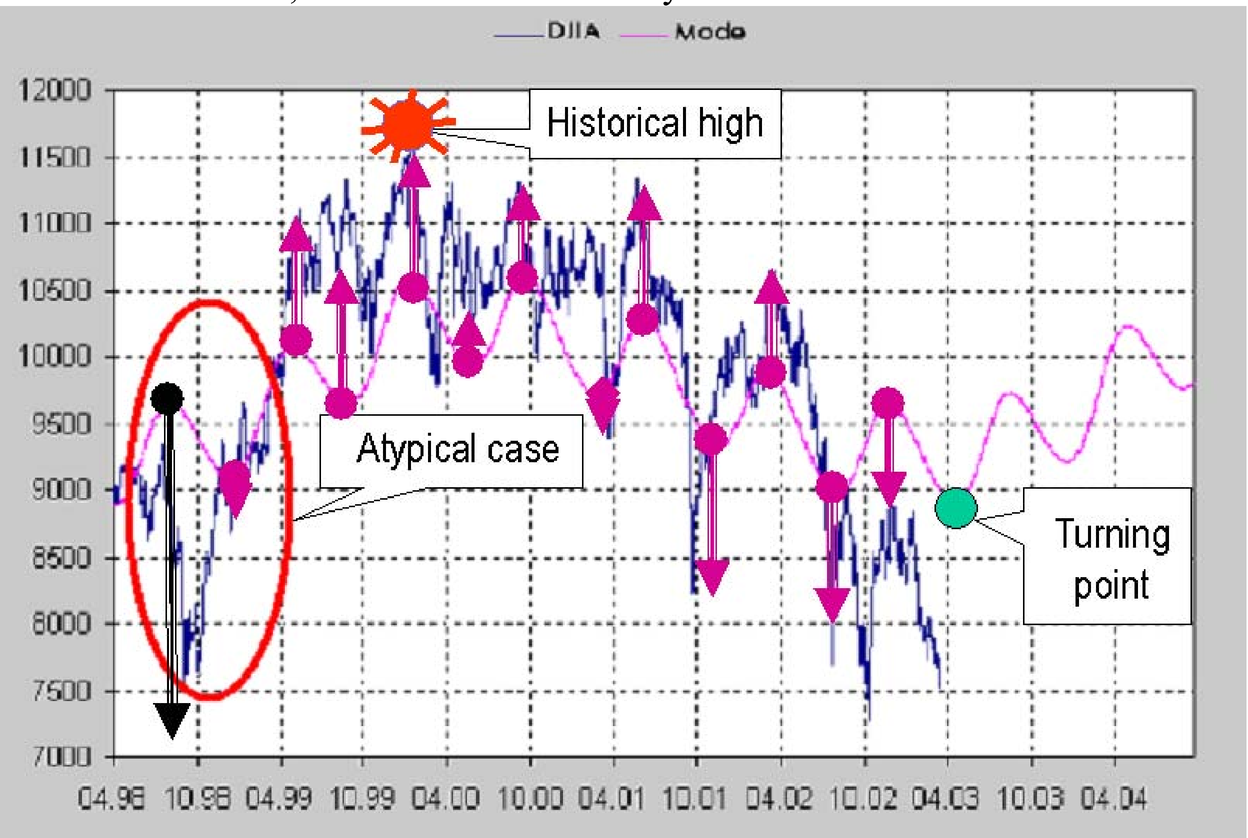


Fig. 4. Analysis of result of the mode selection from DJIA time series.

Disadvantages of the selected mode are also completely obvious: the curve does not approximate the data in the sense of root-mean-square proximity; provided that this shift looks like a systematic error. However, both disadvantages are correctable. As a matter of fact, they are caused by a false attitude in selecting the mode from the process description, which selection is obviously not continual on the time interval being considered. Self-organization of strongly dissipative systems which are a source of DJIA data is a quality sporadically leading them to rather deep kinds of restructuring. The break in the cycle continuity is a result of any restructuring, which fact does not allow us to insist on a hypothesis (assumed by default) that any business cycle is an endlessly renewable continuous curve. In other words, the result is a compromise between the reality and the hypothesis.

Generally speaking, it is the analysis of the business cycle continuity breaks that is the most interesting part of the task on using the model of a 'reaction-diffusion' type in the business cycle analysis; provided that this model is potentially useless therefor. However, this topic lies outside the bounds of this research, though being a natural course in the further development of the related investigations. Thus, two-species Lotka—Volterra model with temporally periodic intermittence of interspecific competitive relationships allows to identify a business cycle in the form of a superposition of the solutions. This solution is obviously defective since it doesn’t imply any unavoidable alterations of a business process over an analyzed interval.